PHYSICAL REVIEW D VOLUME 18, NUMBER 12 15 DECEMBER 1978

Effect of instantons on the heavy-quark potential

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(Received 5 July 1978)

A systematic large-mass expansion of the quark-antiquark effective Hamiltonian due to instantons is developed. The $O(m^{-2})$ spin-spin and spin-orbit contributions are evaluated in the dilute-gas approximation. These can be expressed in terms of differential operators acting on the spin-independent potential. The three-quark effective Hamiltonian (in a color-singlet state) is shown to be a sum of quark-antiquark Hamiltonians. The structure and numerical value of the effective Hamiltonian is discussed.

I. INTRODUCTION

It is by now widely appreciated that the instanton solutions of Yang-Mills field theories describe a new and important aspect of vacuum physics: large semiclassical flustuations of the gauge field associated with tunneling between different realizations of the vacuum.[1] These new fluctuations are qualitatively quite different from the more familiar perturbative zero-point oscillations of the field and yield different physical effects. The physical effects due to instantons have been discussed in detail in Ref. 2 where it was argued that they are responsible for much of the dynamics of quantum chromodynamics (QCD).[2]

One of the effects of instantons is to contribute a new term to the interaction energy of quarks. In Ref. 2 the static (spin-independent) quark-antiquark potential due to instantons was calculated, using the dilute-gas approximation for the analog instanton gas. It was found that as one increased the separation of the quarks this potential increased rapidly, soon becoming larger than the ordinary Coulomb energy which describes quark interactions in the short-distance, asymptotically free, region. In a recent Letter, two of us have given arguments that the instantons are also responsible for a large spin-spin interaction between heavy quarks, which might be of phenomenological importance in the context of heavy-quarkonium spectroscopy.[3] The argument relied on a heuristic extension, to include spin, of the Wilson loop treatment of the instanton-induced interaction of static quarks and did not extract all possible spin-dependent terms in the potential.

In this paper we shall give a systematic treatment of the heavy-quark potential which succeeds in extracting all the interesting spin-dependent effects. Our method allows one, in principle, to expand systematically the effective quark Hamiltonian due to instantons in powers of the inverse quark mass. This is analogous to the Foldy-Wouthuysen[4] transformation, which we shall indeed use. In this paper we shall restrict our attention to the $O(m^{-2})$ spin-dependent terms in the effective Hamiltonian, although higher-order terms could be evaluated. We shall also, just as in many of the applications in Ref. 2, use the dilute-gas approximation, ignoring correlations between neighboring instantons. This means that we include only the effects of instantons smaller than some maximum size, since the density of instantons increases rapidly with scale size, leading to the breakdown of the dilute-gas approximation. In particular we have not included the large-scale fluctuations, due to merons, which we believe responsible for quark confinement.[2] Nevertheless, the dilute-instanton-gas contribution to the heavy-quark potential should provide a qualitative description of tunneling effects at not too large distances, and might be of use in phenomenological descriptions of heavy-quark bound states.

In the following section we shall develop a method for the large-mass expansion of the quark-antiquark potential and determine the leading, $O(m^{-2})$, spin-dependent terms. The end result is a quite elegant form for the spin-dependent interactions which can mainly be expressed (in a fashion reminiscent of the Breit interaction) in terms of the spin-independent potential.

In Sec. III we discuss the precise form of the spin-independent and spin-dependent potentials. For large or small quark separation these can be determined analytically, while for intermediate separations we must evaluate the potential numerically.

Finally, in Sec. IV we discuss the interaction of three heavy quarks in a color-singlet state. Quite remarkably, the three-quark potential turns out to be just the sum of two-body interactions, each equal (including spin terms) to half the quark-antiquark potential.

II. A LARGE-MASS EXPANSION OF THE QUARK HAMILTONIAN

We are interested in those aspects of Euclidean vacuum physics which are accounted for by widely separated, uncorrelated instantons. In such a "dilute-gas approximation" the full effect of the medium on a quark is given by an appropriately weighted average (over location and scale size) of the effect of a single instanton.[2] To be specific, we look at the Euclidean $A_0 = 0$ gauge time history described in Fig. 1. The instanton is in region II and we have placed an artificial sharp cutoff on the instanton E and B fields so that regions I and III are characterized by a vacuum gauge field. The vacuum in region III is topologically inequivalent to I and characterized by the mapping $U(\vec{x})$ of three-space into the group. For a standard SU(2) instanton of scale size ρ centered at $\vec{r}$,

$$U = \exp\{i\pi\vec{\tau}\cdot(\vec{x}-\vec{r})/[(\vec{x}-\vec{r})^2+\rho^2]^{1/2}\} \tag{1}$$

and clearly carries information about the instanton which caused the transition. Eventually we shall see that U contains all the information needed to construct the instanton-generated quark interactions. In the case of SU(3), U has the same form, except that τ matrices are replaced by Gell-Mann λ matrices.

This prescribed, time-dependent gauge field causes the quark wave function to change with time, and to follow this evolution we just use the Dirac Hamiltonian for a fermion in an external non-Abelian gauge field. Since the quark is very massive, it is convenient to use the systematic expansion in powers of m^{-1} provided by the Foldy-Wouthuysen transformation. The transformed Hamiltonian is $H = H_1 + H_2$, where [to $O(1/m^2)$]

$$\begin{aligned} H_1 &= \frac{(\vec{p}-e\vec{A})^2}{2m} + eA_0 , \\ H_2 &= -\frac{e}{2m}\vec{\sigma}\cdot\vec{B} - \frac{e}{4m^2}\vec{\sigma}\cdot\vec{E}\times(\vec{p}-e\vec{A}) - \frac{e}{8m^2}\vec{D}\cdot\vec{E} \\ &\quad - \frac{ie}{8m^2}\vec{\sigma}\cdot(\vec{D}\times\vec{E}) , \end{aligned} \tag{2}$$

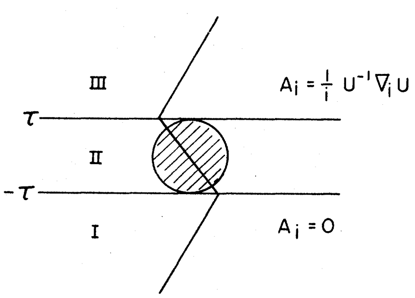

FIG. 1. Representation of the gauge field and heavy-quark trajectory discussed in the text. The hatched area is the region of non-pure-gauge field.

where $A_i = A_i^a\lambda^a/2$, $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}^a\lambda^a/2$, and D_i is the covariant derivative. The Hamiltonian of course acts on color variables and is just the color transcription of the usual Foldy-Wouthuysen transformation.

The quark wave function [a function $\psi(q) = \psi_{a\alpha}(\vec{x})$ of color, two-component spin, and space variables] evolves in time under the action of the propagator

$$S(q_2t_2, q_1t_1) = T\left\{\exp\left[-\int_1^2 dt\, H(t)\right]\right\} . \tag{3}$$

In considering the propagation from time t_1 in region I to time t_2 in region III we use the composition law

$$S(q_2t_2, q_1t_1) = \int dq\, dq'\, S(q_2t_2; q\tau)S(q\tau, q'-\tau)S(q'-\tau, q_1t_1) \tag{4}$$

where of course $\int dq$ means integration over position coordinate and summation over spin and color indices. The propagator through region I is, since $A_i = 0$, just the ordinary Schrödinger propagator $S_0(\vec{x}', -\tau; \vec{x}_1, t_1)$ times unit spin and color matrices. The propagator through region III is just a color gauge transform of the Schrödinger propagator

$$S(q_2t_2, q\tau) = \delta_{\alpha_2\alpha}[U(\vec{x}_2)U^{-1}(\vec{x})]_{\alpha_2\alpha}S_0(\vec{x}_2t_2; \vec{x}\tau) . \tag{5}$$

The color matrix $U^{-1}(\vec{x})$ we combine with the nontrivial region-II propagator $S(q\tau, q'-\tau)$ to obtain the net effect of the instanton on the quark color and spin degrees of freedom. The matrix $U(\vec{x}_2)$ will be absorbed by the next instanton and serves to transform its gauge field back to standard one-instanton form. In summary, the net effect of the instanton on spin and color variables is given by a quasi-local (in time) propagator $U^{-1}(\vec{x})S(q\tau, q'-\tau)$ and propagation between successive instantons is given by the spin- and color-independent Schrödinger propagator.

In region II we expand the propagator of $H = H_1 + H_2$ in powers of H_2. To obtain the leading spin-dependent effects, it suffices to keep the first two terms in the expansion, and we set

$$\begin{aligned} U^{-1}(\vec{x})S(\vec{x}\tau; \vec{y}-\tau) &= U^{-1}(\vec{x})S_1(\vec{x}\tau; \vec{y}-\tau) \\ &\quad - \int_{-\tau}^{\tau} dt \int d^3z\, U^{-1}(\vec{x})S_1(\vec{x}\tau; \vec{z}t) \\ &\quad \times H_2(\vec{z}, t)S_1(\vec{z}t; \vec{y}-\tau) . \end{aligned} \tag{6}$$

In this expression, S_1 is the spin-independent propagator of H_1 and the spin and color indices are implicit. Since H_2 and U are known explicitly, our only problem is to evaluate S_1.

We proceed from the path-integral expression

$$S_1(\vec{x}'t';\vec{x}t)=\int [Dy]\left[\exp\left(-\int_t^{t'} d\tau \frac{m\dot{y}^2}{2} + i\int_x^{x'} d\vec{y}\cdot\vec{A}\right)\right]. \quad (7)$$

Since m is large, the path integral is dominated by the straight-line path between $\vec{x}$ and $\vec{x}'$ and S_1 factorizes,

$$S_1(\vec{x}'t',\vec{x}t) \doteq S_0(\vec{x}'t',\vec{x}t)P\left[\exp\left(ie\int_x^{x'} d\vec{y}\cdot\vec{A}\right)\right], \quad (8)$$

where S_0 is the Schrödinger propagator and the path-ordered integral from $\vec{x}$ to $\vec{x}'$ is taken over the straight-line path. Since we are interested in short-time propagation, S_0 restricts $\vec{x}'$ to be close to $\vec{x}$ and we may characterize the path by a *small* velocity $\vec{v}=(\vec{x}'-\vec{x})/(t'-t)$.

To evaluate the path-ordered integral in (8) it is convenient to transform A from the $A_0=0$ gauge to the "standard" gauge $eA_\mu^a = 2\eta_{a\mu\nu}x_\nu/(x^2+\rho^2)$. This is effected by the time-dependent matrix $U(\vec{x},t) = \exp[i\phi(\vec{x},t)\vec{\tau}\cdot\vec{x}]$, where $\phi(\vec{x},t)=\frac{1}{2}\pi+\tan^{-1}[t/(\vec{x}^2+\rho^2)^{1/2}]$ [note that for $t=+\infty$, $U(\vec{x},t)$ becomes identical to $U(\vec{x})$ as defined in (1)]:

$$U^{-1}(\vec{x}',t')P\left[\exp\left(ie\int_{\vec{x}}^{\vec{x}'} d\vec{y}\cdot\vec{A}\right)\right]U(\vec{x},t) = P\left\{\exp\left[ie\int_{\vec{x}}^{\vec{x}'} dy_\mu \tau_a \eta_{a\mu\nu}y_\nu/(y^2+\rho^2)\right]\right\}.$$

For a straight-line path, $y_\mu(t)=x_\mu+v_\mu t$, and one easily sees that because of the antisymmetry of the η symbol, the new form of the integral is Abelian. Explicitly,

$$U^{-1}(\vec{x}',t')P\left[\exp\left(ie\int_{\vec{x}}^{\vec{x}'} d\vec{y}\cdot\vec{A}\right)\right]U(\vec{x},t) = \exp\left[i\int_t^{t'} d\tau \frac{-\vec{\tau}\cdot\vec{x}+\vec{\tau}\cdot\vec{v}\times\vec{x}}{(x_0+\tau)^2+(\vec{x}+\vec{v}\tau)^2+\rho^2}\right].$$

Since $\vec{v}$ is $O(m^{-1})$, and we are constructing a systematic expansion in powers of m^{-1}, we must expand this integral in powers of $\vec{v}$. In practice we need only keep terms of at most first order in $\vec{v}$, and we find

$$U^{-1}(\vec{x}',t')P\left[\exp\left(ie\int_{\vec{x}}^{\vec{x}'} d\vec{y}\cdot\vec{A}\right)\right]U(\vec{x},t) \doteq \{(1-\vec{v}\cdot\vec{x}\vec{x}\cdot\vec{\nabla})U^{-1}(\vec{x},t')U(\vec{x},t)\} - i\left\{(\vec{x}\cdot\vec{v})\left(\frac{1}{t'^2+\vec{x}^2+\rho^2}-\frac{1}{t^2+\vec{x}^2+\rho^2}\right)(\vec{\tau}\cdot\vec{x})U^{-1}(\vec{x},t')U(\vec{x},t)\right\}. \quad (9)$$

(It should be kept in mind that $\vec{x}$, the quark position coordinate, is measured relative to the instanton.) In the rest of this paper we shall work out the interaction terms due to the term in the first set of curly brackets, eventually showing that they can all be expressed in terms of the spin-independent heavy-quark potential. The term in the second set of curly brackets leads to terms which are not related to the spin-independent potential and, fortunately, quantitatively much less significant. The corresponding heavy-quark potential will be given below, although we shall not present the detailed derivation.[5]

The path-ordered integral in (8) is multiplied by the Schrödinger propagator, $S_0(\vec{x}'t',\vec{x}t)\propto\exp[-m(\vec{x}'-\vec{x})^2/2|t'-t|]$. This means that $\vec{v}$, wherever it appears, may be replaced by a quark momentum operator

$$\vec{v}S_0=\frac{(\vec{x}'-\vec{x})}{(t'-t)}S_0=-\frac{1}{m}\vec{\nabla}_x S_0. \quad (10)$$

Further, since m is large and the time interval over which S_0 is evaluated is finite, we may replace S_0 by $\delta(\vec{x}'-\vec{x})$ and, of course, set $\vec{x}'=\vec{x}$ in the integral. Our final form for the expansion of S_1 to $O(m^{-1})$ is then

$$S_1(\vec{x}'t',\vec{x}t)=U(\vec{x},t')\{\delta(\vec{x}-\vec{x}')+[(1/m)(\vec{x}\times\vec{\nabla})U^{-1}(\vec{x},t')U(\vec{x},t)]\cdot\vec{\nabla}_x\delta(\vec{x}-\vec{x}')\}U^{-1}(\vec{x},t). \quad (11)$$

We now are in a position to evaluate (6). Consider first the term $U(\vec{x})S_1(\vec{x}\tau,\vec{y}-\tau)$. Since $U(\vec{x},-\tau)=1$ and $U(\vec{x},\tau)=U(\vec{x})$, (11) implies that

$$U^{-1}(\vec{x})S_1(\vec{x}\tau,\vec{y}-\tau)=[U^{-1}(\vec{x})+(1/m)(\vec{x}\times\vec{\nabla})U^{-1}(\vec{x})\cdot\vec{\nabla}_x]\delta(\vec{x}-\vec{y}). \quad (12)$$

Next consider the term in (6) in which H_2 appears. We will first deal with the $O(m^{-1})$ part of H_2, $-(e/2m)\vec{\sigma}\cdot\vec{B}$. Since we wish to evaluate (6) to $O(m^{-2})$, we need to keep the $O(m^{-1})$ terms in S_1. The rather complicated expression which results can be simplified considerably if we remember that $U^{-1}(\vec{x},t)\vec{B}(\vec{x},t)U(\vec{x},t) = \bar{\vec{B}}(\vec{x},t)$, the instanton magnetic field in the "standard" gauge, and that $\bar{\vec{B}}(\vec{x},t)$ actually depends only on $\vec{x}^2$ so that it commutes with the operator $\vec{x}\times\vec{\nabla}$. This allows us to rearrange the various derivatives and recast the expression in the form

$$\int_{-\tau}^{\tau} dt\, U^{-1}(\vec{x})\frac{e}{2m}\vec{\sigma}\cdot\bar{\vec{B}}(\vec{x},t)\delta(\vec{x}-\vec{y})+\left[\int_{-\tau}^{\tau} dt\frac{(\vec{x}\times\vec{\nabla})}{m}U^{-1}(\vec{x})\frac{e}{2m}\vec{\sigma}\cdot\bar{\vec{B}}(\vec{x},t)\right]\cdot\vec{\nabla}\delta(\vec{x}-\vec{y}). \quad (13)$$

Since $\vec{B}$ is the instanton magnetic field in the $A_0=0$ gauge, $\vec{B}=-\vec{E}=-\dot{\vec{A}}$, and the time integration may be done explicitly converting (13) to

$$\frac{i}{2m}(\vec{\sigma}\cdot\vec{\nabla})U^{-1}(\vec{x})\delta(\vec{x}-\vec{y})+\frac{i}{2m^2}(\vec{x}\times\vec{\nabla})(\vec{\sigma}\cdot\vec{\nabla})U^{-1}(\vec{x})\cdot\vec{\nabla}\delta(\vec{x}-\vec{y}). \tag{14}$$

Finally, we must include the $O(m^{-2})$ part of H_2. To the order we are interested in, we may replace S_1 in (6) by $\delta(\vec{x}-\vec{z})$. We also keep only the $\vec{\sigma}\cdot(\vec{E}\times\vec{p})$ term in H_2 because the gauge field actually satisfies the Yang-Mills equations. Again, since in the $A_0=0$ gauge $\vec{E}=\dot{\vec{A}}$, we may do the time integration, and the end result is

$$-\frac{i}{4m^2}[\vec{\sigma}\times\vec{\nabla}U^{-1}(\vec{x})]\cdot\vec{\nabla}\delta(\vec{x}-\vec{y}). \tag{15}$$

We now combine (12)–(15) to obtain the net effect of the instanton on the heavy-quark wave function. Propagation across the time slice occupied by the instanton causes the transformation

$$\psi(\vec{x})\to\tilde{\psi}(\vec{x})=U^{-1}(\vec{x})\psi(\vec{x})+\frac{1}{m}[\vec{x}\times\vec{\nabla}U^{-1}(\vec{x})]\cdot\vec{\nabla}\psi(\vec{x})+\frac{i}{2m}[\vec{\sigma}\cdot\vec{\nabla}U^{-1}(\vec{x})]\psi(\vec{x})$$
$$+\frac{i}{2m^2}\{(\vec{x}\times\vec{\nabla})[\vec{\sigma}\cdot\vec{\nabla}U^{-1}(\vec{x})]\}\cdot\vec{\nabla}\psi(\vec{x})-\frac{i}{4m^2}[\vec{\sigma}\times\vec{\nabla}U^{-1}(\vec{x})]\cdot\vec{\nabla}\psi(\vec{x}). \tag{16}$$

This can be more compactly written as

$$\tilde{\psi}(\vec{x})=[O(\vec{x},\vec{p},\vec{\sigma})U^{-1}(\vec{x})]\psi(\vec{x}), \tag{17}$$

$$O(\vec{x},\vec{p},\vec{\sigma})=1-\frac{i}{m}\vec{L}\cdot\vec{\nabla}-\frac{i}{2m}\vec{\sigma}\cdot\vec{\nabla}-\frac{1}{2m^2}\vec{L}\cdot\vec{\nabla}\vec{\sigma}\cdot\vec{\nabla}+\frac{1}{4m^2}\vec{\sigma}\cdot(\vec{p}\times\vec{\nabla}), \tag{18}$$

where of course $\vec{L}=\vec{x}\times\vec{p}$ and we have made a distinction between $\vec{\nabla}$, which operates only on $U^{-1}(\vec{x})$, and $\vec{p}$, which operates on the $\vec{x}$ coordinate of ψ. We note here that the corresponding expression for an antiquark is obtained by replacing U^{-1} in (17) by $[U]^T$. It should also be said that we have, in order to simplify formulas, at various stages dropped some $O(m^{-2})$ terms in O which are not of direct phenomenological interest: some are spin-independent and some would only appear in the $\theta\neq 0$ vacuum, giving rise to P- or T-violating effects.

Now we are ready to construct the effective Hamiltonian for the $q\bar{q}$ system. The wave function is $\psi_{a_1\alpha_1,a_2\alpha_2}(\vec{x}_1,\vec{x}_2)$ where a, α, and $\vec{x}$ are color, spin, and space indices, respectively, and the effect of the instanton is given by the tensor product of single-quark expressions.

$$\tilde{\psi}_{a_1\alpha_1,a_2\alpha_2}(\vec{x}_1\vec{x}_2)=O(1)_{\alpha_1\alpha_1'}O(2)_{\alpha_2\alpha_2'}U^{-1}(x_1)_{a_1a_1'}\times U^T(x_2)_{a_2a_2'}\psi_{a_1'\alpha_1'a_2'\alpha_2'}(\vec{x}_1,\vec{x}_2). \tag{19}$$

The effective Hamiltonian is identified by $-H\psi=\tilde{\psi}-\psi$, which amounts to replacing $U^{-1}(1)\otimes U(2)^T$ by $[U^{-1}(1)\otimes U(2)^T-1\otimes 1]$. Since we are in practice interested only in color-singlet states we may perform a trace over color variables, obtaining (for SU_3)

$$H^{\text{singlet}}=O(\vec{x}_1p_1\sigma_1)O(\vec{x}_2p_2\sigma_2)\{-\tfrac{1}{3}\mathrm{tr}[U(\vec{x}_1)U^{-1}(\vec{x}_2)-1]\}. \tag{20}$$

The last step is to integrate over the instanton spatial position and scale size with the appropriate density function $D(\rho)$, obtaining

$$H^{\text{singlet}}=O(\vec{x}_1p_1\sigma_1)O(\vec{x}_2p_2\sigma_2)V(\vec{x}_1-\vec{x}_2), \tag{21}$$

where

$$V(\vec{x}_1-\vec{x}_2)=-2\int\frac{d\rho}{\rho^5}D(\rho)\times\int d^3r\tfrac{1}{3}\mathrm{tr}[U(\vec{x}_1-\vec{r})U^{-1}(\vec{x}_2-\vec{r})-1] \tag{22}$$

is the spin-independent heavy-quark potential which has been discussed previously.[2] The factor of 2 in (22) comes from adding the effect of anti-instantons. We shall see that in a $\theta=0$ vacuum, the two contributions are equal. There is one minor complication which must be mentioned. In the product $O(1)O(2)$, there are terms depending explicitly on the quark-instanton coordinate difference $x-x_I$. This dependence comes entirely from the terms involving $\vec{L}$ and is at most linear in $x-x_I$ if $O(1)O(2)$ is evaluated to $O(m^{-2})$. Thus, in integrating over the instanton coordinates we appear to encounter a new potential $\bar{V}$, defined by

$$(x_1-x_2)_i\bar{V}(x_1-x_2)=-\int\frac{d\rho}{\rho^5}D(\rho)\int d^3r(x_1-r)_i\times\{\tfrac{1}{3}\mathrm{tr}[U(\vec{x}_1-\vec{r})U^{-1}(\vec{x}_2-\vec{r})-1]\}+(\text{instanton}\to\text{anti-instanton}).$$

Fortunately, straightforward manipulations show that $\bar{V}=\frac{1}{2}V$, V being the spin-independent potential defined in (22). Therefore, in the expansion of $O(1)O(2)$ appearing in (21) we must understand that $\vec{L}_1=\frac{1}{2}(\vec{x}_1-\vec{x}_2)\times\vec{p}_1$ and $\vec{L}_2=-\frac{1}{2}(\vec{x}_1-\vec{x}_2)\times\vec{p}_2$.

Our main concern here is not the precise form of V, but the fact that spin-dependent corrections to it can be expressed directly in terms of its derivatives. The expansion of $O(1)O(2)$ contains terms of $O(m^{-1})$ and $O(m^{-2})$. The terms of $O(m^{-1})$ are explicitly CP violating and survive only if we construct a $\theta\neq 0$ vacuum. The reason is that if we go through the same arguments, replacing instanton by anti-instanton, the only effect is to change the sign of the $O(1/m)$ terms in O. Then when instanton (anti-instanton) effects are weighted by $e^{i\theta}$ ($e^{-i\theta}$) and the two terms summed, the CP-violating terms appear multiplied by $\sin\theta$. Since experiment constrains θ to be essentially zero, we may restrict our attention to the $O(1/m^2)$ terms. They are of two varieties: spin-spin and spin-orbit (the latter defined rather loosely as a term involving spin of one particle and momentum of the other). The spin-spin term is

$$\begin{aligned}H_{ss}&=-\frac{1}{4m_1m_2}(\vec{\sigma}_1\cdot\vec{\nabla}_1)(\vec{\sigma}_2\cdot\vec{\nabla}_2)V(\vec{x}_1-\vec{x}_2)\\&=\frac{1}{4m_1m_2}\sigma_1\cdot\sigma_2\,\tfrac{1}{3}\nabla^2V(\vec{x}_1-\vec{x}_2)\\&\quad+\frac{1}{4m_1m_2}\sigma_i^1\sigma_j^2(\nabla_i\nabla_j-\tfrac{1}{3}\delta_{ij}\nabla^2)V(\vec{x}_1-\vec{x}_2)\,.\end{aligned}\tag{23}$$

(We have separated the tensor force from the spin-spin force.) These expressions are superficially quite similar to the usual Breit potentials, but, since V is not a Coulomb potential the r dependence is quite different. The spin-orbit term is more complicated,

$$\begin{aligned}H_{so}=\Big[&-\frac{1}{2m_1^2}\vec{L}_1\cdot\vec{\nabla}_1\vec{\sigma}_1\cdot\vec{\nabla}_1+\frac{1}{4m_1^2}\vec{\sigma}_1\cdot(\vec{p}_1\times\vec{\nabla}_1)\\&-\frac{1}{2m_1m_2}\vec{L}_1\cdot\vec{\nabla}_1\vec{\sigma}_2\cdot\vec{\nabla}_2+(1\leftrightarrow 2)\Big]V(\vec{x}_1-\vec{x}_2)\,,\end{aligned}\tag{24}$$

where, as mentioned in the discussion of (21), we must take

$$\vec{L}_1=\tfrac{1}{2}(\vec{x}_1-\vec{x}_2)\times\vec{p}_1\,,\quad \vec{L}_2=-\tfrac{1}{2}(\vec{x}_1-\vec{x}_2)\times\vec{p}_2\,.$$

At this point we note that the neglected terms in Eq. (9) can be shown[5] to give rise to a spin-orbit term of the form

$$H'_{so}=-\left(\frac{1}{m_1^2}\vec{\sigma}_1\cdot\vec{L}_1+\frac{1}{m_2^2}\vec{\sigma}_2\cdot\vec{L}_2\right)\tilde{V}(\vec{x}_1-\vec{x}_2)\,,$$

where $\tilde{V}$ is a new potential function unrelated to V. A rough estimate[5] suggests that H'_{so} is small (≈10% of H_{so}).

III. THE STRUCTURE OF THE EFFECTIVE HAMILTONIAN

We have shown that to order $1/m^2$ the heavy-quark effective Hamiltonian can be expressed in terms of the spin-independent potential $V(\vec{x})$ given in Eq. (22). In this section we shall discuss the form and magnitude of this potential, as well as the limitations of the dilute-gas approximation.

The spin-independent potential is given by

$$V(x)=+2\int\frac{d\rho}{\rho^2}D(\rho)W\left(\frac{x}{\rho}\right),$$

where the dimensionless potential $W(x/\rho)$, due to an instanton of size ρ, is defined by

$$\begin{aligned}W\left(\frac{x}{\rho}\right)&=-\frac{1}{3\rho^3}\int d^3r\,\mathrm{tr}[U(\vec{x}-\vec{r})U^{-1}(-\vec{r})-1]\\&=-\frac{1}{\rho^3}\frac{2}{3}\int d^3r\left\{\cos\frac{\pi r}{(r^2+\rho^2)^{1/2}}\cos\frac{\pi|\vec{x}-\vec{r}|}{[(r-x)^2+\rho^2]^{1/2}}\right.\\&\qquad\left.+\frac{\vec{r}\cdot(\vec{r}-\vec{x})}{r|\vec{r}-\vec{x}|}\sin\frac{\pi r}{(r^2+\rho^2)^{1/2}}\sin\frac{\pi|\vec{x}-\vec{r}|}{[(x-r)^2+\rho^2]^{1/2}}-1\right\}.\end{aligned}\tag{25}$$

This integral cannot be analytically evaluated for all values of x. However we can find analytic expressions in the limits $x\to 0$ or $x\to\infty$. For small $z=x/\rho$,

$$\begin{aligned}W(Z)\underset{Z\to 0}{\sim}&\,Z^2\int d^3r\left[\frac{\pi^2}{9(r^2+1)^3}\frac{2\sin^2\pi\left(\frac{r^2}{1+r^2}\right)^{1/2}}{9r^2}\right]+O(Z^4)\\&=\frac{\pi^3}{9}[\pi/4-4J_1(2\pi)]Z^2+O(Z^4)\\&=5.635Z^2+O(Z^4)\,.\end{aligned}\tag{26}$$

Thus

$$V(x)\underset{x\to 0}{\sim}11.27x^2\int\frac{d\rho}{\rho^3}D(\rho)+O(x^4)\,.\tag{27}$$

We then can deduce from Eqs. (23) and (24) the small-distance behavior of the spin-dependent terms,

$$H_{ss}=\frac{5.635}{m_1m_2}(\vec{\sigma}_1\cdot\vec{\sigma}_2)\int\frac{d\rho}{\rho^3}D(\rho)\tag{28}$$

and

$$H_{so}=\left[-\frac{1}{m_1{}^2}(\vec{x}_{12}\times\vec{p}_1)\cdot\vec{\sigma}_1-\frac{1}{2m_1m_2}(\vec{x}_{12}\times\vec{p}_2)\cdot\vec{\sigma}_1+\frac{1}{m_2{}^2}(\vec{x}_{12}\times\vec{p}_2)\cdot\vec{\sigma}_2+\frac{1}{2m_1m_2}(\vec{x}_{12}\times\vec{p}_1)\cdot\vec{\sigma}_2\right]11.27\int\frac{d\rho}{\rho^3}D(\rho)\,. \tag{29}$$

For large values of x, $W(x/\rho)$ approaches a constant. This is to be expected since the background instanton field falls rapidly (in singular gauge) at large distances, so that when the quarks are separated by a distance much greater than the instanton size only one quark at a time is affected by the instanton, leading to a mass renormalization for large x/ρ.

For large x/ρ, we can expand W as follows:

$$W\left(\frac{x}{\rho}\right)=\frac{1}{3\rho^3}\int d^3r\,\mathrm{Tr}\{[1+U(\vec{x}-\vec{r})]+[1+U^{-1}(-\vec{r})]\}-\frac{1}{3\rho^3}\int d^3r\,\mathrm{Tr}\{[1+U(\vec{x}-\vec{r})][1+U^{-1}(-\vec{r})]\}$$
$$\underset{x\gg\rho}{\longrightarrow}\frac{4}{3\rho^3}\int d^3r\left[1+\cos\frac{\pi r}{(r^2+\rho^2)^{1/2}}\right]-\frac{2}{3\rho^3}\int d^3r\,r\hat{r}\cdot(\hat{r}-\hat{x})\sin\frac{\pi r}{(r^2+\rho^2)^{1/2}}\sin\frac{\pi(r-x)}{[(r-x)^2+\rho^2]^{1/2}}\,. \tag{30}$$

Therefore,

$$W\left(\frac{x}{\rho}\right)\underset{x\gg\rho}{\longrightarrow}\tfrac{8}{9}\pi^3[-\pi J_0(\pi)-J_1(\pi)]-\frac{2\pi^3\rho}{3x}+O\left(\left(\frac{\rho}{x}\right)^2\right) \tag{31}$$

and

$$V(x)\underset{x\to\infty}{\longrightarrow}37\int\frac{d\rho}{\rho^2}D(\rho)-\frac{4\pi^3}{3x}\int\frac{d\rho}{\rho}D(\rho)+O\left(\frac{1}{x^2}\right)\,. \tag{32}$$

As noted above, the first term corresponds to a mass renormalization of $18.5\int(d\rho/\rho^2)D(\rho)$ per quark. The second term is an asymptotic contribution to the Coulomb potential. The standard perturbative Coulomb interaction for a color-singlet quark-antiquark state is $\frac{4}{3}(g^2/4\pi)/x$, so that this additional term can be interpreted as a renormalization of the QCD coupling constant due to instantons,

$$\frac{g^2}{8\pi^2}\to\frac{g^2}{8\pi^2}\left[1+\frac{\pi^2}{2}\int\frac{d\rho}{\rho}\left(\frac{8\pi^2}{g^2}\right)D(\rho)\right]. \tag{33}$$

This is identical with the coupling-constant renormalization derived in Ref. 2 by evaluating the gluon propagator in a dilute-gas approximation. Thus we see that instantons induce a mass and coupling renormalization. With reasonable cutoffs on the ρ integration these are large but finite, thus showing that instantons do not confine.

For intermediate values of x the integral in Eq. (25) cannot be performed analytically. We therefore present a numerical evaluation of $W(x/\rho)$ in Figs. 2 and 3, from which the spin-dependent parts of the Hamiltonian can be derived.[6] For example, in Fig. 4 we plot $W_{\sigma_1\sigma_2}$ defined by

$$H_{\sigma_1\sigma_2}=\frac{1}{2m_1m_2}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\int\frac{d\rho}{\rho^2}D(\rho)W_{\sigma_1\sigma_2}\left(\frac{x}{\rho}\right),$$
$$W_{\sigma_1\sigma_2}(z)=\frac{1}{3}\left[2\frac{W'(z)}{z}+W''(z)\right],$$

and in Fig. 5 we plot W_T, defined by

$$H_{\rm tensor}=-\frac{1}{4m_1m_2}\left[\vec{\sigma}_1\cdot\hat{x}\vec{\sigma}_2\cdot\hat{x}-\frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{3}\right]2\int\frac{d\rho}{\rho^2}W_T\left(\frac{x}{\rho}\right),$$
$$W_T(z)=\frac{W'(z)}{z}-W''(z)\,. \tag{34}$$

To obtain finally a potential which could be compared with experiment (in the form, say, of the low-lying charmonium spectrum) we must do an integration over instanton scale sizes. Needless to say, our formulas only make sense for scale sizes where the dilute-gas approximation is valid. According to Ref. 2 this means that we must impose an upper cutoff ρ_c on our integrals, where ρ_c

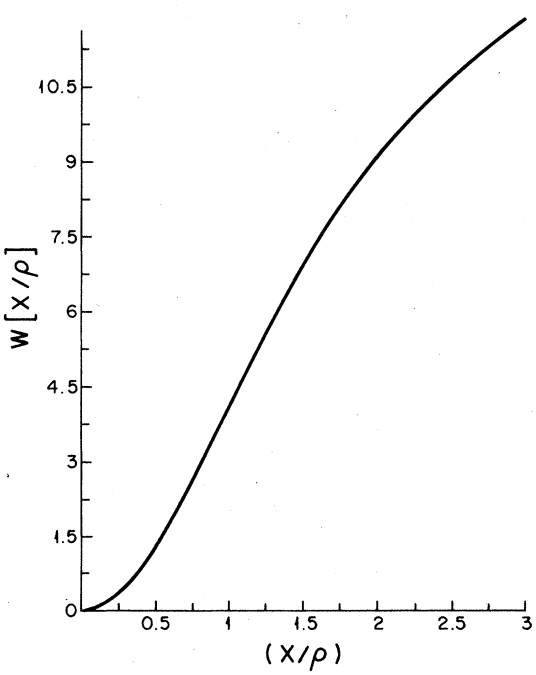

FIG. 2. The spin-independent heavy-quark potential due to instantons of a definite scale size.

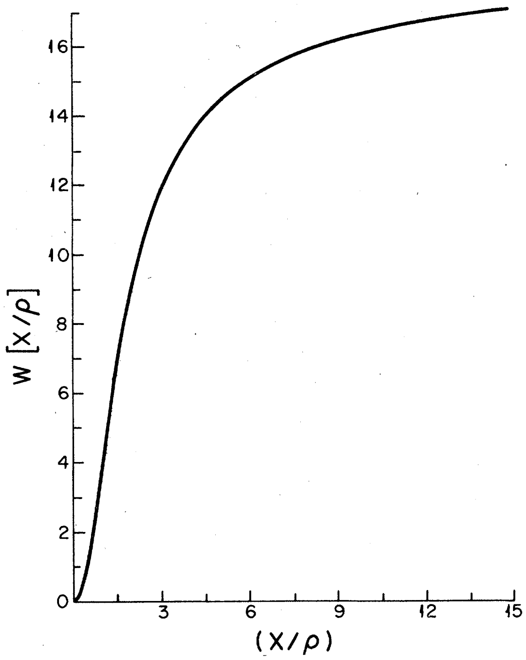

FIG. 3. The same function as Fig. 2, but displaced on a larger scale in the variable x/ρ so as to show the asymptotic approach to a constant potential.

is such that the effective instanton coupling $8\pi^2/g^2(\rho_c)$ is about 15. Since ρ_c is not too much different from the confinement scale of the theory, we would expect that the physics of confinement would in any case cut off the effect on our poten-

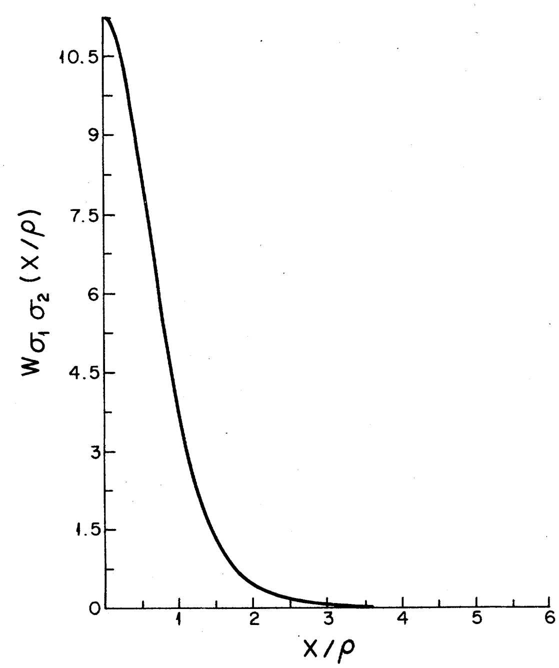

FIG. 4. The spin-spin potential generated by instantons of a definite scale size.

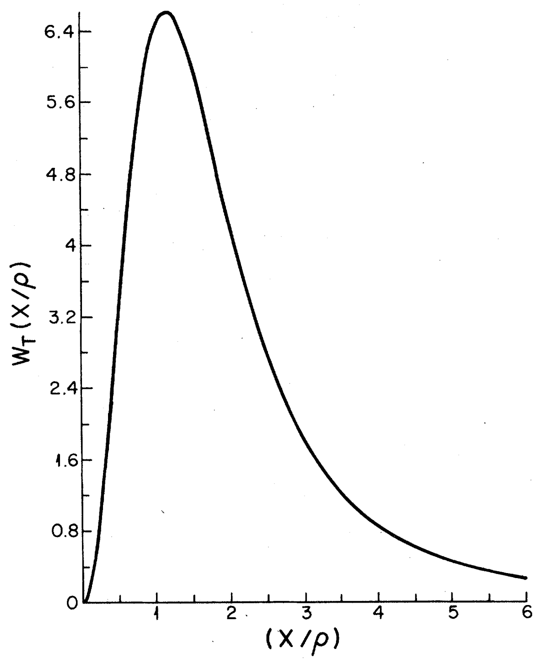

FIG. 5. The tensor part of the spin-dependent potential generated by instantons of a definite scale size.

tials of fluctuations on a scale much bigger than ρ_c. Now, a typical integral over scale sizes is $\int_0^{\rho_c} d\rho\, \rho^{-3} D(\rho)$, the quantity which governs the short-distance behavior of the spin-independent potential (27) and the spin-spin potential (28). Because of the factor ρ^{-3}, large scales are somewhat suppressed, but since our choice of ρ_c lies in a region where $D(\rho)$ is rapidly rising, the cutoff integral depends very strongly on the choice of cutoff.

What this means is that if we want to put a precise number on, say, the curvature of the spin-independent potential at the origin [Eq. (27)], we really have to understand the physics of the confinement mechanism which operates at scale sizes equal to or greater than ρ_c. Presumably the cutoff dilute-gas contribution gives a rough estimate of the size of such effects, but it would be difficult to claim that it is accurate enough for serious phenomenological applications. This is characteristic of all the applications we have studied and distinguishes instanton effects from perturbative asymptotic-freedom effects. The latter may be both suppressed and rendered calculable by looking at phenomena on a sufficiently small scale: by looking at large momentum one concentrates on the effect of small fluctuations. Instanton effects can indeed be rendered small by going to large momenta; however, their magnitude always depends on all scales up to the confinement scale simultaneously and may therefore only be qualitatively characterized until one has a more detailed knowledge of the confinement mechanism.

IV. THE THREE-QUARK POTENTIAL

Using the methods developed in Sec. II we can evaluate the instanton-generated effective Hamiltonian for any color-singlet system of quarks. In addition to $q\bar{q}$ (meson) systems we might be interested in qqq (baryon) systems. In the nonrelativistic approximation such a system will be described by a three-body wave function $\psi_{a_1\alpha_1, a_2\alpha_2, a_3\alpha_3}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ where a, α, and $\vec{x}$ are the color, spin, and space indices of the quarks. As we have seen, the effect of a single instanton is given by the tensor product of the single-quark operators $O(\vec{x}, \vec{p}, \vec{\sigma})U(\vec{x})$. Thus for a baryon state

$$\tilde{\psi}_{a_1\alpha_1, a_2\alpha_2, a_3\alpha_3} = O(1)_{\alpha_1\alpha_1'}O(2)_{\alpha_2\alpha_2'}O(3)_{\alpha_3\alpha_3'}U(\vec{x}_1)_{a_1a_1'}U(\vec{x}_2)_{a_2a_2'}U(\vec{x}_3)_{a_3a_3'}\psi_{a_1'\alpha_1', a_2'\alpha_2', a_3'\alpha_3'}(\vec{x}_1, \vec{x}_2, \vec{x}_3) . \tag{35}$$

The effective three-quark Hamiltonian arising from an instanton of given size and position is then obtained by setting $-H\psi = \tilde{\psi} - \psi$. We pick out the color-singlet state of three quarks by means of the projection $(1/\sqrt{6})\epsilon_{a_1a_2a_3}$, thereby deriving

$$H_{qqq}^{\text{singlet}} = \tfrac{1}{6}\, O(x_1p_1\sigma_1)O(x_2p_2\sigma_2)O(x_3p_3\sigma_3)\left[\epsilon_{a_1a_2a_3}\epsilon_{a_1'a_2'a_3'}U(x_1)_{a_1a_1'}U(x_2)_{a_2a_2'}U(x_3)_{a_3a_3'} - 3\right] . \tag{36}$$

Thus, as in the case of the $q\bar{q}$ potential, the effective Hamiltonian can be constructed from a spin-independent potential $V(\vec{x}_1, \vec{x}_2, \vec{x}_3)$. After integrating over instanton positions and scale sizes and adding instantons and anti-instantons, we obtain

$$H_{qqq}^{\text{singlet}} = O(x_1p_1\sigma_1)O(x_2p_2\sigma_2)O(x_3p_3\sigma_3)V(\vec{x}_1\vec{x}_2\vec{x}_3) , \tag{37}$$

where

$$V(\vec{x}_1, \vec{x}_2, \vec{x}_3) = -2\int \frac{d\rho}{\rho^5} D(\rho) \int d^3r \int d\hat{R}\, \tfrac{1}{6}\left[\epsilon_{a_1a_2a_3}\epsilon_{a_1'a_2'a_3'}U^R(\vec{x} - \vec{r})_{a_1a_1'}U^R(\vec{x}_2 - \vec{r})_{a_2a_2'}U^R(\vec{x}_3 - \vec{r})_{a_3a_3'} - 6\right] , \tag{38}$$

where R is an SU(3) rotation matrix which specifies the group orientation of the instanton.

Now we note an important special property of SU(3) instantons—namely that they are embeddings in SU(3) of SU(2) instantons. Thus

$$U^R = R\left[\exp\left(\frac{i\pi\tau^a(x^a - r^a)}{[(\vec{x} - \vec{r})^2 + \rho^2]^{1/2}}\right)\right]R^{-1} = R\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}R^{-1} , \quad U = \exp\left(\frac{i\pi\vec{\tau}\cdot(\vec{x} - \vec{r})}{[(\vec{x} - \vec{r})^2 + \rho^2]^{1/2}}\right) . \tag{39}$$

Since Eq. (38) is invariant under the global rotation R, it is clear that

$$\epsilon_{a_1a_2a_3}\epsilon_{a_1'a_2'a_3'}U^R(\vec{x}_1 - \vec{r})_{a_1a_1'}U^R(\vec{x}_2 - \vec{r})_{a_2a_2'}U^R(\vec{x}_3 - \vec{r})_{a_3a_3'} - 6 = \sum_{ij}\left\{\mathrm{Tr}\,U(\vec{x}_i - \vec{r})\,\mathrm{Tr}\,U(\vec{x}_j - \vec{r}) - \mathrm{Tr}\,U(\vec{x}_i - \vec{r})U(\vec{x}_j - \vec{r})\right\} - 2 . \tag{40}$$

But for such two-by-two special unitary matrices the following identity holds, $\mathrm{Tr}U_1\,\mathrm{Tr}U_2 - \mathrm{Tr}U_1U_2 = \mathrm{Tr}U_1U_2^\dagger = \mathrm{Tr}U_2U_1^\dagger$, therefore the three-body potential of Eq. (38) can be written as a sum of three terms, each of which is recognizable as half of the two-body quark-antiquark potential derived in Sec. II,[7]

$$V(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \tfrac{1}{2}\sum_{i>j=1}^{3} V_{q\bar{q}}(\vec{x}_i - \vec{x}_j) . \tag{41}$$

The factor of $\frac{1}{2}$ in this expression is strange, but the following argument will easily convince the reader of its validity. Let two of the quarks sit on top of each other, i.e., set $x_1 = x_2$. The two quarks then behave exactly like an antiquark so that $V_{3q}(x_1 = x_2, x_3)$ should equal $V_{q\bar{q}}(x_1 - x_3)$, in agreement with Eq. (42). Note that if we were dealing with an SU(N) gauge group the analogous equation would be

$$V_{Nq}(x_1, x_2, \ldots, x_N) = \frac{1}{N-1}\sum_{i>j=1}^{N} V_{q\bar{q}}(x_i - x_j) . \tag{42}$$

Since the operator $O(x, p, \sigma)$ reduces to the identity when acting on an x-independent function we can therefore express the $3q$ Hamiltonian as a sum (with a factor of $\frac{1}{2}$) of two-body Hamiltonians each of which is identical to the quark-antiquark Hamiltonian

$$H_{qqq}^{\text{singlet}} = \tfrac{1}{2}\sum_{i>j=1}^{3} O(x_i, p_i, \sigma_i)O(x_j, p_j, \sigma_j) \times V(x_i - x_j) . \tag{43}$$

This remarkable simplicity is a consequence of the fact that the instanton in SU(3) is embedded within a single SU(2) subgroup. Thus a given background instanton field only affects two colors at a time. In a baryon state, which is composed of three different colored quarks, only two of the quarks are affected by a given instanton. When we average over instanton group orientations the energy is then a sum of two-body interaction energies. This additivity will be a feature of any contribution to the potential due to a semiclassical configuration which is embedded in an SU(2) subgroup, for example, meron pairs treated in the dilute-gas approximation. However, the group-orientation-dependent instanton–anti-instanton interactions[2] will induce true many-body forces.

ACKNOWLEDGMENTS

The work of C. G. C. was supported in part by the National Science Foundation under Grant No. NSF-PHY78-01221. The work of D. J. G. was supported in part by the U. S. Department of Energy under Grant No. PHY77-20612. The work of R. D., F. W., and A. Z. was supported in part by the U. S. Department of Energy on Grant No. EY76-S-02-2220. F. W. and A. Z. acknowledge the support of the A. P. Sloan Foundation.

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*On leave from Princeton University, 1977-78.

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[5]A forthcoming publication by C. Carvalho will provide the detailed derivation.

[6]We would like to thank L. Yaffe for performing the numerical evaluation of W.

[7]Note that this expression for the spin-independent potential could be derived from a generalization of the Wilson loop argument. One simply considers the expectation value of the operator

$$O=\epsilon_{a_1 a_2 a_3}\epsilon_{a'_1 a'_2 a'_3}\prod_{i=1}^{3}(^{U}L_i)_{a_i a'_i},$$

where $U_L=T\{\exp(i\int A^\mu dx_\mu)\}$ and L_i $(i=1, \ldots, 3)$ are three paths in Euclidean space which run from the point at which the quarks are created to the point at which they are annihilated. If these paths are taken to be straight timelike lines (for a time T) at positions $\vec{x}_i$, then at $T\to 0$ the vacuum expectation value of O will approach $\exp[-TV(\vec{x}_1,\vec{x}_2,\vec{x}_3)]$.